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Time reversal for drifted fractional Brownian motion with Hurst index $H > 1/2$

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Abstract

Let X be a drifted fractional Brownian motion with Hurst index $H > 1/2$. We prove that there exists a fractional backward representation of X, i.e. the time reversed process is a drifted fractional Brownian motion, which continuously extends the one obtained in the theory of time reversal of Brownian diffusions when $H = 1/2$. We then apply our result to stochastic differential equations driven by a fractional Brownian motion.

Key words: Fractional Brownian motion. Time reversal. Malliavin Calculus.

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1 Introduction

Time reversal for diffusion processes driven by a Brownian motion (Bm in short) has already been studied by several authors in the Markovian case (9; 17; 10) and by Föllmer (7) in the Markovian and non Markovian case. The question can be summed up as to know whether the time reversed process is again a diffusion and how to compute its reversed drift and its reversed diffusion coefficient. Different approaches have been proposed. Haussmann and Pardoux (9) tackle this problem by means of weak solutions of backward and forward Kolmogorov equations; Pardoux (17) bases its approach on the enlargement of a filtration. In both cases, the reversed drift, the reversed diffusion coefficient and the Brownian motion driving the reversed diffusion are explicitely identified. Millet, Nualart and Sanz (10) use the integration by parts from Malliavin Calculus and obtain under mild assumptions the expressions of the reversed drift and diffusion coefficient.

These approaches find their roots in Föllmer's work (7) for a class of drifted Brownian motions. He gives, under a finite entropy condition, a formula for the reversed drift of a non Markovian diffusion with a constant diffusion coefficient. He deeply uses the relation between drifts and the forward and backward derivatives introduced by Nelson (12) in his dynamical theory of Brownian diffusions. From a dynamical point of view, Nelson's operators are fundamental tools as regards Brownian diffusions. Based on these operators, it is possible to define an operator which extends classical differentiation from smooth deterministic functions to classical diffusion processes and which allows to give stochastic analogue to standard differential operators (see (3)). Unfortunately these operators fail to exist for a simple fractional Brownian motion (fBm in short) when $H \neq 1/2$ (*cf* Proposition 10 of this paper).

The question one may then address is to know if we can obtain a drifted fBm for the time reversed process of a drifted fBm, which extends the one obtained in the Brownian case. Despite the non existence of Nelson's derivatives for a fBm, we prove that the answer to this question is positive by using the transfer principle and Föllmer's formula in the non Markovian Brownian diffusion case.

Let us explain more precisely our result in the case of a fractional diffusion. This example is further described in Section 5. Let $1/2 \leqslant H < 1$ and $(Y_t^H)_{t\in[0,T]}$ be the solution of the stochastic differential equation

$$dY_t^H = u(Y_t^H)dt + dB_t^H,$$

where the function u is bounded and has bounded first derivative and $(B^H)_H$ is a family of fBm transfered from a unique Bm $B^{1/2}$. When $H = 1/2$, we know (see (17)) that the time reversed process $\overline{Y}^{1/2}$ defined by $\overline{Y}_t^{1/2} = Y_{T-t}^{1/2}$ is again a diffusion process given by

$$d\overline{Y}_t^{1/2} = \left(-u(\overline{Y}_t^{1/2}) + \partial_x \log p_{T-t}(\overline{Y}_t^{1/2})\right) dt + d\widehat{B}_t^{1/2} \ ,$$

where $p_t(\cdot)$ is the density of the law of the process $Y^{1/2}$ at time t and $\widehat{B}^{1/2}$ is a Brownian motion with respect to the filtration generated by $\overline{Y}^{1/2}$.

Our main result extends this formula in the following way: we are able to find both a drift process $\widehat{u}^H$ and a fBm $\widehat{B}^H$ such that

$$d\overline{Y}_t^H = \widehat{u}_t^H dt + d\widehat{B}_t^H$$

and such that the following convergences hold in $L^1(\Omega)$:

$$\lim_{H\downarrow 1/2} \widehat{B}_t^H = \widehat{B}_t^{1/2}, \quad \text{and}$$
$$\lim_{H\downarrow 1/2} \int_0^t \widehat{u}_s^H ds = \int_0^t \left(-u(\overline{Y}_s^{1/2}) + \nabla \log p_{T-s}(\overline{Y}_s^{1/2})\right) ds.$$

Why this result may be interesting? In the Brownian case, the drift and the time reversed drift are respectively the forward Nelson derivative and minus the backward Nelson derivative which are actually notions of mean velocities. Although these objects do not exist in the fractional case, the drift of a fractional diffusion can be always thought as a forward velocity with respect to the driving process. From this point of view, our structure theorem for the time reversed fractional diffusion gives a backward velocity which is explicit from the initial drift and coherent in the following sense: this quantity is a "continuous" extension in H of the well known notion of the Wiener case. This prevents from non relevant decompositions. Our method of construction is natural but non trivial. We finally mention that an other notion of velocity based on stochastic derivatives with respect to some "differentiating" σ-fields, can be found in (4).

Our paper is organized as follows. In Section 2 we present some preliminary definitions and results about fractional Brownian motion. We recall in Section 3 Föllmer's strategy to tackle the time reversal problem for Brownian diffusion both in the Markovian and the non Markovian case. In Section 4, we state our main result about the existence of a reversed drift for a drifted fBm with $1/2 < H < 1$ which "continuously" extends the one obtained in the Wiener case. Moreover, we prove that Nelson's derivatives are inappropriate tools for a fBm. Section 5 is devoted to the application of our result to fractional diffusions. In Section 6, we discuss the way to construct operators extending Nelson's operators in the fractional case. For the sake of completeness, we finally include in an appendix the proofs of some crucial results from Föllmer (7).

2 Notations and preliminary results

We briefly recall some basic facts about stochastic integration with respect to a fBm. One refers to (14; 1; 2) for further details. Let $B^H = (B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm in short) with Hurst parameter $H > 1/2$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We mean that B^H is a centered Gaussian process with the covariance function $E(B_s^H B_t^H) = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \ . \tag{1}$$

If $H = 1/2$, then B^H is clearly a Brownian motion (Bm in short). From (1), one can easily see that $E|B_t^H - B_s^H|^p = E|G|^p.|t-s|^{pH}$ for any $p \geqslant 1$, where G is a centered Gaussian variable with variance 1. So the process B^H has $\alpha-$Hölder continuous paths for all $\alpha \in (0, H)$.

Spaces of deterministic integrands

We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]} \ , \ \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

Then the scalar product between two elements φ and ψ of $\mathcal{E}$ is given by

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u \, dr du. \tag{2}$$

When $H > 1/2$, the space $\mathcal{H}$ contains $L^{\frac{1}{H}}(0,T;\mathbb{R})$ but its elements may be distributions. However, Formula (2) holds for $\varphi, \psi \in L^{\frac{1}{H}}(0,T;\mathbb{R})$.

The mapping

$$\mathbf{1}_{[0,t]} \mapsto B_t^H$$

can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with B^H. We denote this isometry by

$$\varphi \mapsto B^H(\varphi) \ .$$

The covariance kernel $R_H(t,s)$ introduced in (1) can be written as

$$R_H(t,s) = \int_0^{s \wedge t} K_H(s,u) K_H(t,u) du \ ,$$

where $K_H(t,s)$ is the square integrable kernel defined by

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du \tag{3}$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-1/2)}\right)^{1/2}$, for $s < t$ (β denotes the Beta function). We set $K_H(t,s) = 0$ if $s \geqslant t$.

We introduce the operator $\mathcal{K}_H^* : \mathcal{E} \to L^2(0,T;\mathbb{R})$ defined by:

$$\mathcal{K}_H^* \left(\mathbf{1}_{[0,t]}\right) = K_H(t,.).$$

It holds that (see (14)) for any $\varphi, \psi \in \mathcal{E}$

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{K}_H^* \varphi, \mathcal{K}_H^* \psi \rangle_{L^2(0,T;\mathbb{R})} = E\left(B^H(\varphi) B^H(\psi)\right)$$

and then $\mathcal{K}_H^*$ provides an isometry between the Hilbert space $\mathcal{H}$ and $L^2(0,T;\mathbb{R})$.

We finally denote by $\mathcal{K}_H$ the operator defined by

$$\left\{ \begin{array}{ccc} L^2(0,T;\mathbb{R}) & \longrightarrow & \mathcal{K}_H\left(L^2(0,T;\mathbb{R})\right) \\ \dot{h} & \longmapsto & (\mathcal{K}_H \dot{h})(t) := \int_0^t K_H(t,s) \dot{h}(s) ds \ . \end{array} \right.$$

The space $\mathcal{K}_H\left(L^2(0,T;\mathbb{R})\right)$ is the fractional version of the Cameron-Martin space. In the case of a classical Brownian motion $K_H(t,s) = \mathbf{1}_{[0,t]}(s)$.

Transfer principle

In this work, we shall often use the link between the stochastic integration of deterministic integrand with respect to the fBm and with respect to a Wiener process which is naturally associated with B^H. This correspondence is usually called the transfer principle.

The process $W = (W_t)_{t\in[0,T]}$ defined by

$$W_t = B^H\big((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0,t]})\big) \tag{4}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)dW_s \ . \tag{5}$$

For any $\varphi \in \mathcal{H}$, it holds that

$$B^H(\varphi) = W\left(\mathcal{K}_H^*\varphi\right) \ . \tag{6}$$

Fractional Calculus

In order to describe more precisely some spaces related to the integration of deterministic elements of $\mathcal{H}$, we need further notations.

Let $a, b \in \mathbb{R}$, $a < b$. For any $p \geqslant 1$, we denote by $L^p(a,b)$ the usual Lebesgue spaces of functions on $[a,b]$ and $|.|_{L^p(a,b)}$ the associated norm.
Let $f \in L^1(a,b)$ and $\alpha > 0$. The left fractional Riemann-Liouville integral of f of order α is defined for almost all $x \in (a,b)$ by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)dy \ ,$$

where Γ denotes the Euler function. This integral extends the classical integral of f when $\alpha = 1$.

Let $I_{a+}^\alpha(L^p)$ the image of $L^p(a,b)$ by the operator I_{a+}^α. If $f \in I_{a+}^\alpha(L^p)$ and $\alpha \in (0,1)$, then for almost all $x \in (a,b)$, the left-sided Riemann-Liouville derivative of f of order α is defined by its Weyl representation

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\right)\mathbf{1}_{(a,b)}(x) \ , \tag{7}$$

and $I_{a+}^\alpha\big(D_{a+}^\alpha f\big) = f$.

In this framework, the operator $\mathcal{K}_H$ has the following properties. First, the square integrable kernel K_H is given by (see (6)):

$$K_H(t,s) = \Gamma(H+\frac{1}{2})^{-1}\ (t-s)^{H-\frac{1}{2}}\ F\left(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{s}\right) \ ,$$

where F is the Gauss hypergeometric function. The operator $\mathcal{K}_H$ is an isomorphism from $L^2(0,T)$ onto $I_{0^+}^{H+\frac{1}{2}}\big(L^2(0,T)\big)$ and it can be expressed as follows when $H > 1/2$:

$$\mathcal{K}_H h = I_{0^+}^1 s^{H-\frac{1}{2}} I_{0^+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h \tag{8}$$

where $h \in L^2(0,T)$. The inverse operator $\mathcal{K}_H^{-1}$ is given by

$$\mathcal{K}_H^{-1}\varphi = s^{H-\frac{1}{2}} D_{0^+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H}\varphi' \tag{9}$$

for all $\varphi \in I_{0^+}^{H+\frac{1}{2}}\big(L^2(0,T)\big)$.

A fundamental remark for the sequel is that the expression (8) shows that $\mathcal{K}_H h$ is an absolutely continuous function when $H \geqslant 1/2$. In this case, we then set

$$(\mathcal{O}_H h)(s) := \left(\frac{d}{dt}\mathcal{K}_H\right)(h)(s) = s^{H-\frac{1}{2}}(I_{0^+}^{H-\frac{1}{2}} r^{\frac{1}{2}-H} h)(s). \tag{10}$$

We will need in the sequel the following technical lemma related to the fractional operator $\mathcal{O}_H$.

Lemma 1. *Set $H_0 > 1/2$. Let $f \in L^1(0,T)$ and assume that f satisfies the condition $\int_0^T |f(u)|u^{1/2-H_0}du < +\infty$. Then for all $t \in [0,T]$ and $H \in (\frac{1}{2}, H_0)$,*

$$\int_0^t |(\mathcal{O}_H f)(s)|\, ds \leqslant C(H_0)\int_0^t |f(u)|u^{1/2-H_0}du \tag{11}$$

and

$$\lim_{H\downarrow 1/2}(\mathcal{K}_H f)(t) = \int_0^t f(u)du. \tag{12}$$

Proof. Fix $t \in (0,T)$. We use (3) and Fubini theorem to write when $H > 1/2$:

$$(\mathcal{K}_H f)(t) = c_H \int_0^t f(u)u^{1/2-H}\int_u^t s^{H-1/2}(s-u)^{H-3/2}dsdu,$$

where the constant is the one given in the definition of K_H: $c_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-1/2)}\right)^{1/2}$.

But for all $H \in (1/2, H_0)$,

$$\left|f(u)u^{1/2-H}\int_u^t s^{H-1/2}(s-u)^{H-3/2}ds\right| \leqslant$$
$$|f(u)|\left(u^{1/2-H_0}\vee 1\right)\left(t^{H_0-1/2}\vee 1\right)\frac{(t-u)^{H_0-1/2}\vee 1}{H-1/2}\ .$$

Since $\frac{c_H}{(H-1/2)} \to 1$ when $H \downarrow 1/2$, we get

$$c_H\left|f(u)u^{1/2-H}\int_u^t s^{H-1/2}(s-u)^{H-3/2}ds\right| \leqslant$$
$$C|f(u)|\left(u^{1/2-H_0}\vee 1\right)\left(t^{H_0-1/2}\vee 1\right)^2$$

then the inequality (11). Moreover, using $u^{H-1/2} \leqslant s^{H-1/2} \leqslant t^{H-1/2}$, the following limit holds:

$$\lim_{H\downarrow 1/2} c_H \int_u^t s^{H-1/2}(s-u)^{H-3/2}ds = 1.$$

The hypothesis on the function f allows us to apply the dominated convergence theorem, which yields the convergence (12). $\square$

Sample path properties

We finally need the following Lemma about path regularity of processes parameterized by H.

Lemma 2. *Let* $(x_t^H)_{t\in[0,T]}$ *be a process such that* $E|x_t^H - x_s^H|^p \le c_p|t-s|^{pH}$ *for some* $p \ge 5$ *and* $H \in [1/2,1)$. *Then*

$$|x_t^H - x_s^H| \leqslant C_{p,T}\ \xi_{p,H}\ |t-s|^{H-2/p}, \tag{13}$$

where ξ *is a positive random variable such that* $\sup_{H\in[1/2,1)} E\xi_{p,H}^p \leqslant 1$.

We essentially do the same computations as in the proof of Lemma 7.4 in (16) but we give precisions about the dependence on the parameter H of the quantities involved in (13), especially the fact that the random variable ξ_H has moments of order $q = 1,...,5$ independent of the parameter H. This will play an important role in our application to stochastic differential equations driven by a fBm.

Of course, this result applies to a fBm since $(E|B_t^H - B_s^H|^p)^{1/p} = c_p|t-s|^H$ where $c_p = (E|G|^p)^{1/p}$ with G a centered Gaussian variable.

Proof. With $\psi(u) = u^p$ and $p(u) = u^H$ in Lemma 1.1 of (8), the Garsia-Rodemich-Rumsey inequality reads as follows:

$$|x_t^H - x_s^H| \leqslant 8\int_0^{|t-s|} \left(\frac{4B}{u^2}\right)^{1/p} Hu^{H-1}du,$$

where the random variable B is

$$B = \int_0^T \int_0^T \frac{|x_t^H - x_s^H|^p}{|t-s|^{pH}} dtds\ .$$

We denote by $\xi_{p,H} = B^{1/p}$ and we have

$$\begin{aligned}|x_t^H - x_s^H| &\leqslant 8\ 4^{1/p}\ \xi_{p,H} \int_0^{|t-s|} H\ u^{H-1-2/p}du \\ &\leqslant 8\ 4^{1/p}\ \xi_{p,H}\ \tfrac{H}{H-2/p}\ |t-s|^{H-2/p}\end{aligned}$$

and since $p \geqslant 5$ and $H \in [1/2,1)$, we have

$$|x_t^H - x_s^H| \leqslant 80\ 4^{1/p}\ \xi_{p,H}|t-s|^{H-2/p}.$$

Moreover

$$E\xi_{p,H}^p \leqslant \int_0^T \int_0^T \frac{E|x_t^H - x_s^H|^p}{|t-s|^{pH}} dtds \leqslant c_pT^2.$$

We set $C_{p,T} = 80 \times 4^{1/p}\, T^2 c_p$ and the result is proved. □

3 Reminder of time reversal on the canonical probability space

In this section, we recall fundamental results on time reversal on the Wiener space. We essentially use the tools and the results stated by Föllmer in (7).

Let us denote by $(X_t)_{t\in[0,T]}$ the coordinate process defined on the canonical probability space $(\Omega_*, (\mathcal{F}_t), \mathbb{W}_*)$ where $\Omega_* = \mathcal{C}([0,T])$ is the space of real valued continuous functions on $[0,T]$ endowed with the supremum norm, $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration (generated by the coordinate process X) and $\mathbb{W}_*$ the Wiener measure. Let $\mathbb{W}$ be an equivalent measure to $\mathbb{W}_*$. By the Girsanov Theorem, there exists an adapted process $(b_t)_{t\in[0,T]}$ satisfying

$$\int_0^T |b_t|^2 dt < \infty \quad \mathbb{W}\text{ -a.s.}$$

such that the process defined by

$$W_t = X_t - \int_0^t b_s ds \tag{14}$$

is a Bm under $\mathbb{W}$.

We say that $\mathbb{W}$ has finite entropy with respect to $\mathbb{W}_*$ if

$$H(\mathbb{W}|\mathbb{W}_*) = E_{\mathbb{W}}\left(\log \frac{d\mathbb{W}}{d\mathbb{W}_*}\right) < \infty. \tag{15}$$

According to Proposition 2.11 p.122 in (7), this condition is equivalent to the following finite energy condition (with respect to $\mathbb{W}$):

Definition 3. *A process $(b_t)_{t\in[0,T]}$ is said to have finite energy on $[0,\tau]$, $\tau \leqslant T$, with respect to a measure $\mathbb{Q}$ if*

$$E_{\mathbb{Q}}\left[\int_0^\tau |b_s|^2 ds\right] < \infty.$$

When no confusion is possible, we will omit the measure.

We denote by $\widehat{\mathbb{W}} = \mathbb{W} \circ R$ the image of $\mathbb{W}$ under pathwise time reversal R on $\mathcal{C}([0,T])$ defined by

$$X_t \circ R = X_{T-t}.$$

The following result of Föllmer (*cf* Lemma 3.1 in (7)) ensures the existence of this reversed drift:

Lemma 4. *If $\mathbb{W}$ has finite entropy with respect to $\mathbb{W}_*$, then there exists an adapted process $(\widehat{b}_t)_{t\in[0,T]}$ with finite energy on $[0,\tau]$, for any $\tau < T$, such that*

$$\widehat{W}_t = X_t - \int_0^t \widehat{b}_s \; ds \;, \quad 0 \leqslant t \leqslant T$$

is a $(\mathcal{F}_t, \widehat{\mathbb{W}})$-Bm.

Notice that the reversed drift has only finite energy on $[0,\tau]$ for any $\tau < T$ and not on the entire time interval $[0,T]$.

Föllmer starts from a finite entropy measure and thus produces a finite energy drift. He then works with several measures and their "reversal". Nevertheless for our main result stated in the next section, it is important to work with a unique probability measure.

We stress on the obvious fact that the reversed process $X_t \circ R = X_{T-t}$ under $\mathbb{W}$ has the same law than the process X_t under $\widehat{\mathbb{W}}$. So by considering the reversed processes $\overline{X}_t := X_t \circ R$, $\widehat{b}_t \circ R$, $\widehat{W}_t \circ R$ and the filtration $R(\mathcal{F}_t) = \sigma\{X_s, T-t \leqslant s \leqslant T\}$ we can rewrite Lemma 4 in terms of a unique probability measure.

We start from a drifted Bm defined on a probability space and we have to impose a condition on its drift to obtain the corresponding measure (the one from which Föllmer starts). We choose the Novikov condition, namely

$$E\left[\exp \int_0^T b_s^2 ds\right] < \infty. \tag{16}$$

The finite energy of the drift is then a straightforward consequence of (16).

We now state a fundamental result of Föllmer on which our main result (Theorem 9) is based on.

Theorem 5. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space and let X be a drifted Bm defined by*

$$X_t = x + \int_0^t b_s ds + W_t \tag{17}$$

where $(W_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t)$-Bm and the drift $(b_t)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted and satisfies the Novikov condition (16). We denote by $(\widehat{\mathcal{F}}_t)_{t\in[0,T]}$ the filtration generated by the reversed process $(\overline{X}_t)_{t\in[0,T]}$ defined by

$$\overline{X}_t = X_{T-t}\ .$$

Then $\overline{X}$ is a drifted Bm: there exists a $(\widehat{\mathcal{F}}_t)$-adapted process $(\hat{b}_t)_{t\in[0,T]}$ and a $(\widehat{\mathcal{F}}_t)$-Bm $(\widehat{W}_t)_{t\in[0,T]}$ such that

$$\overline{X}_t = \overline{X}_0 + \int_0^t \hat{b}_s ds + \widehat{W}_t.$$

The process $(\hat{b}_t)_{t\in[0,T]}$ has finite energy on $[0,\tau]$, $0 < \tau < T$, and belongs to $L^p(\Omega \times (0,T))$ for any $p \in (1,2)$.

The original Föllmer's result only mentions that the time reversed drift has finite energy on $[0,\tau]$, $0 < \tau < T$. But it turns out to be also in $L^p(\Omega \times (0,T))$ for any $p \in (1,2)$. This fact induces significant simplifications in the proof of Lemma 11. We give the proof of this theorem in Appendix A.1.

Under the finite entropy condition (15), one can express the drift process $(b_t)_{t\in[0,T]}$ appearing in (14) in terms of Nelson derivative of the process X.

Definition 6. *Let X be a $\mathcal{F}_t-$adapted process and $(\mathcal{G}_t)_{t\in[0,T]}$ be a decreasing filtration with respect to which X is adapted. The forward and backward Nelson derivative of X are respectively defined for almost all $t \in (0,T)$ as*

$$\mathbf{D}_+ X_t = \lim_{h\downarrow 0} E\left(\frac{X_{t+h} - X_t}{h}\middle| \mathcal{F}_t\right) \quad \textit{in } L^p(\Omega),$$
$$\mathbf{D}_- X_t = \lim_{h\downarrow 0} E\left(\frac{X_t - X_{t-h}}{h}\middle| \mathcal{G}_t\right) \quad \textit{in } L^p(\Omega),$$

for some $p \geqslant 1$, when these limits exist.

The above expressions turn out to be the key point for the explicit computation of the reversed drift of the diffusion X both in Markovian and non Markovian case.

We henceforth work with $\mathcal{G}_t := \widehat{\mathcal{F}}_{T-t} = \sigma(X_s; t \leqslant s \leqslant T)$. We might refer to the filtrations $(\mathcal{F}_t)_{t\in[0,T]}$ and $(\mathcal{G}_t)_{t\in[0,T]}$ as respectively the past of X and the future of X.

The drift process b of X as well as the drift $\widehat{b}$ of $\overline{X}$ have the following expression in terms of Nelson derivatives.

Proposition 7. *Let X be of the form $dX_t = b_t dt + dW_t$ where the process $(b_t)_{t\in[0,T]}$ is $\mathcal{F}_t-$adapted and has finite energy on $[0,T]$. We denote by $\widehat{b}$ the drift of $\overline{X}$ (its existence is ensured by Theorem 5). Then for all $t \in (0,T)$,*

$$\mathbf{D}_+ X_t = b_t, \tag{18}$$

$$\mathbf{D}_- X_t = -\widehat{b}_{T-t}. \tag{19}$$

Proof. We refer to Proposition 2.5 p.121 in (7) for a detail proof of (18).

Writing $X_t - X_{t-h} = -(\overline{X}_{T-t+h} - \overline{X}_{T-t})$ and using that $\widehat{b}$ has finite energy on $[0,\tau]$ for all $\tau \in (0,T)$, we deduce that

$$\hat{b}_{T-t} = -\lim_{h\downarrow 0} E\left(\frac{X_t - X_{t-h}}{h}\middle| \hat{\mathcal{F}}_{T-t}\right) \qquad \text{in } L^2(\Omega)\ .$$

and (19) is then proved. □

We now recall Föllmer's formula of the reversed drift $\widehat{b}$. This result will be useful in the last part of the paper when we apply our main result to a fractional diffusion process.

To this end, we notice that since the drift satisfies the Novikov condition (16), the Girsanov Theorem insures us that $(X_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t, \mathbb{Q})$-Bm under the probability measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = G$ where

$$G = \exp\left(-\int_0^T b_s dW_s - 1/2\int_0^T b_s^2 ds\right)\ . \tag{20}$$

We use the classical notations of Malliavin Calculus with respect to the Bm X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$. More precisely, we denote D the Malliavin derivative operator, $\mathbb{D}^{1,2}$ its domain and $\mathbb{L}^{1,2}$ the Hilbert space which is isomorphic to $L^2([0,T]; \mathbb{D}^{1,2})$ as it is defined in Definition 1.3.2 in (13).

Theorem 8. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space and let X be a drifted Bm which writes:*

$$X_t = x + \int_0^t b_s ds + W_t$$

where $(W_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t)$-Bm and the drift process $(b_t)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted and satisfies the Novikov condition (16) and the following conditions:

1. $(b_t)_{t\in[0,T]} \in \mathbb{L}^{1,2}$,

2. *for almost all t, the process $(D_t b_s)_{s\in[0,T]}$ is Skorohod integrable,*

3. *there exists a version of the process $\left(\int_0^T D_t b_s dW_s\right)_{t\in[0,T]}$ in $L^2([0,T]\times\Omega; dt\otimes d\mathbb{Q})$.*

Then the reversed drift reads

$$\widehat{b}_{T-t} = -E\left(b_t + \frac{1}{t}\left(W_t - \int_0^t \int_v^T D_v b_s dW_s dv\right) + \int_t^T D_t b_s dW_s \,\middle|\, \widehat{\mathcal{F}}_{T-t}\right). \tag{21}$$

For the sake of completeness, we also give the proof of Theorem 8 in Appendix A.2.

4 Existence of a continuously extended drift for the time reversed drifted fBm for $H > 1/2$

In this section, we consider a family of fBm $(B^H)_{H\in[1/2,1)}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ transfered from a unique Bm W: for all $H > 1/2$

$$B_t^H = \int_0^t K_H(t,s) dW_s.$$

4.1 Main result

We are interested in drifted processes of the form

$$Y_t^H = y + \int_0^t u_s^H ds + B_t^H, \tag{22}$$

where $y \in \mathbb{R}$ and $(u_t^H)_{t\in[0,T]}$ is an $\mathcal{F}_t-$adapted process. A natural question is to know if the time reversed drifted fBm Y^H is again a drifted fBm, which extends the one obtained in the Brownian case. We mean that if the formula is parameterized by H, we have to recover the results stated in Theorem 8 for the drifted Brownian motion defined by (22) when $H = \frac{1}{2}$:

$$Y_t^{1/2} = y + \int_0^t u_s ds + W_t. \tag{23}$$

We show in the next theorem that the reversed process of the drifted fBm Y^H can be driven by a fBm $\widehat{B}^H$ which is related to the Wiener process $\widehat{B}^{1/2}$ driving the reversed process $\overline{Y}^{1/2}$ (defined by $\overline{Y}_t = Y_{t-t}$) in the sense that $\lim_{H\to 1/2} \widehat{B}_t^H = \widehat{B}_t^{1/2}$ in $L^1(\Omega)$. We will also give a relation between the drifts of $\overline{Y}^H$ and the one of $\overline{Y}^{1/2}$.

We will need the following conditions:

(i) for all $H \in [1/2, 1)$, the process $b^H := \mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)$ satisfies the Novikov condition (16),

(ii) There exists $H_0 > 1/2$ such that

a) the process $(u_t^H)_{t\in[0,T]}$ has Hölder continuous trajectories of order $H_0-1/2$, and there exists $\eta > H-\frac{1}{2}$ such that $E|u_t^H - u_s^H| \leqslant c|t-s|^\eta$

b) There exists $H_0 > 1/2$ such that

$$\sup_{H\in[1/2,H_0]} E\left[\int_0^T |b_t^H|^2 dt\right] < +\infty.$$

c) For almost all $t \in [0,T]$,

$$\lim_{H\downarrow\frac{1}{2}} E|u_t^H - u_t| \to 0 \ .$$

Remark that the condition **(i)** is also given for $H=1/2$. This implies that the process $(u_t)_{t\in[0,T]}$ also satisfies the Novikov condition. Moreover, this condition implies that $b^H \in \mathbb{L}^2(\Omega\times[0,T])$. Applying the operator $\mathcal{K}_H$ we deduce that the drift has the special form

$$\int_0^t u_s^H ds = \int_0^t K_H(t,s) b_s^H ds \ .$$

Besides, this fact will be used in Theorem 9 via Lemma 11 below.

We can now state the main result of our work.

Theorem 9. *Given a family of processes $(u_t^H)_{t\in[0,T]}$ which satisfies conditions* ***(i)*** *and* ***(ii)****, let $(Y_t^H)_{t\in[0,t]}$ be a family of processes such that*

$$Y_t^H = y + \int_0^t u_s^H ds + B_t^H \ .$$

Then there exists a family of continuous processes $\left(\widehat{u}^H\right)_{H\geqslant 1/2}$ and a family of fBm $(\widehat{B}^H)_{H\geqslant 1/2}$ such that the time reversed process $(\overline{Y}_t^H)_{t\in[0,T]}$ defined by $\overline{Y}_t^H = Y_{T-t}^H$ satisfies

$$\overline{Y}_t^H = \overline{Y}_0^H + \int_0^t \widehat{u}_s^H ds + \widehat{B}_t^H, \tag{24}$$

with for all $t\in(0,T)$

$$\lim_{H\downarrow 1/2} \widehat{B}_t^H = \widehat{B}^{1/2} = \widehat{W}_t \quad \text{ in } L^1(\Omega), \tag{25}$$

$$\lim_{H\downarrow 1/2} \int_0^t \widehat{u}_s^H ds \to \int_0^t \widehat{u}_s ds \quad \text{ in } L^1(\Omega)$$

where $\widehat{W}$ and $\widehat{u}$ are respectively the $\mathcal{F}^{\overline{Y}^{1/2}}$-adapted Bm (reversed drift) produced by the time reversal of the process $Y^{1/2}$ defined by

$$Y_t^{1/2} = y + \int_0^t u_s ds + W_t. \tag{26}$$

Remark that the assumptions **(i)** insure us that the results on time reversal for the drifted Bm

$$X_t^H = y + \int_0^t b_s^H ds + W_t$$

with $b_\cdot^H := \mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)$ are valid. Actually, this assumption is sufficient to construct the reversed drift and the reversed fBm for our drifted fBm Y^H.

The assumptions of **(ii)** are used in order to prove that the drift we construct satisfies a kind of robustness with respect to the parameter H as it is explained in the following subsection.

4.2 Remarks and questions

4.2.1 Continuous extension as a structure constraint

The property (25) is important if we want to formalize the idea that the reversed formula (24) has to extend the reversed one in the classical Wiener case. In that sense, we might say that our formula is a continuously extended formula of the Wiener case. One might think about this extension as the "commutativity" of the following informal diagram:

$$\begin{array}{ccc}
dY_t^H = u_t^H dt + dB_t^H & \xrightarrow[H\downarrow 1/2]{\lim} & dY_t^{1/2} = u_t dt + dW_t \\
\Big\downarrow R & & \Big\downarrow R \\
d\overline{Y}_t^H = \widehat{u}_t^H dt + d\widehat{B}_t^H & \xrightarrow[H\downarrow 1/2]{\lim} & d\overline{Y}_t^{1/2} = \widehat{u}_t dt + d\widehat{W}_t \ ,
\end{array}$$

where R is the time reversal procedure based on the transfer principle. When $H = 1/2$, there is no transfer to do.

This notion of continuously extension plays its hole part if we consider the naive and trivial decomposition

$$\overline{Y}_t^H = \overline{Y}_0^H + \int_{T-t}^T u_s^H ds + B_{T-t}^H - B_T^H. \tag{27}$$

The process $\widetilde{B}_t^H := B_{T-t}^H - B_T^H$ is a fBm (it is centered Gaussian process and has R_H as covariance function), but (27) is not a formula which extends in our sense the one obtained in the Wiener case since

$$\lim_{H\downarrow 1/2} \widetilde{B}_t^H = W_{T-t} - W_T \neq \widehat{W}_t \quad \text{ in } L^1(\Omega).$$

Actually, the decomposition $\overline{Y}_t^{1/2} = \overline{Y}_0^{1/2} + \int_{T-t}^T u_s ds + W_{T-t} - W_T$ is not the Doob-Meyer decomposition of the semi-martingale $\overline{Y}^{1/2}$ with respect to its natural filtration $\mathcal{F}^{\overline{Y}^{1/2}}$.

Although the decomposition of Theorem 9 is an extension of the classical Wiener formula, we have lost the structure of adaptation with respect to $\mathcal{F}^{\overline{Y}}$: in the example of the next section we can show for instance that the fBm $\widehat{B}^H$ produced by our theorem is not adapted with respect to $\mathcal{F}^{\overline{B}^H}$ by showing that the drift is not adapted.

4.2.2 Non existence of Nelson derivatives

Moreover, one may wonder if we can hope to obtain a drifted reversed fBm using Nelson's derivatives. Unfortunately, Nelson's derivatives are inappropriate as an operator acting on drifted fBm thanks to the following proposition.

Proposition 10. *Set $H \neq 1/2$. The limit*

$$\lim_{h \downarrow 0} E\left(\frac{B^H_{t+h} - B^H_t}{h} \middle| \mathcal{F}_t \right)$$

exists neither as an element in $L^p(\Omega)$ for any $p \in [1, \infty)$ nor as an almost sure limit.

Proof. Let $p \in [1, \infty)$. The process defined by $W_t = B^H\big((\mathcal{K}^*_H)^{-1}(\mathbf{1}_{[0,t]})\big)$ is a $\mathcal{F}_t$−Bm. Then we immediately deduce that

$$E\left[B^H_{t+h} - B^H_t \,|\mathcal{F}_t\right] = \int_0^t \left(K_H(t+h,s) - K_H(t,s)\right) dW_s := hZ_h.$$

We fix $t \in (0,T)$. We note that $(Z_h)_{h>0}$ is a centered Gaussian process. The variance of Z_h is given by:

$$\sigma_h^2 = \int_0^t \left| \frac{K_H(t+h,s) - K_H(t,s)}{h} \right|^2 ds.$$

If Z_h converges in $L^p(\Omega)$ or almost surely to a random variable Z when h tends to 0, then Z_h converges in law to Z, and we know that Z is centered Gaussian variable with variance $\sigma^2 = \lim_{h\downarrow 0} \sigma_h^2$. But we shall prove that σ_h^2 does not converge when h tends to 0. Indeed, since $t \mapsto K_H(t,s)$ is differentiable with

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-3/2} \ ,$$

we have:

$$\lim_{h\downarrow 0} \left(\frac{K_H(t+h,s) - K_H(t,s)}{h} \right)^2 = c_H^2 \left(\frac{t}{s}\right)^{2H-1} (t-s)^{2H-3}.$$

Therefore we deduce from Fatou Lemma that

$$\begin{aligned}
\liminf_{h\downarrow 0} \int_0^t & \left(\frac{K_H(t+h,s) - K_H(t,s)}{h} \right)^2 ds \\
&\geqslant \int_0^t \liminf_{h\downarrow 0} \left(\frac{K_H(t+h,s) - K_H(t,s)}{h} \right)^2 ds \\
&= c_H^2 \int_0^t \left(\frac{t}{s}\right)^{2H-1} (t-s)^{2H-3} ds \\
&= +\infty \ .
\end{aligned}$$

So we conclude that when h tends to 0, Z_h converges neither in $L^p(\Omega)$ nor almost surely. □

Some related results are extended and studied in more details in (4).

4.2.3 The case $H < 1/2$

The techniques we have developed may provide a analogous theorem in the case $H < 1/2$, where moreover the formulas and the study of the operators $\mathcal{K}_H^{-1}$ are more tractable. However, we lost the structure of a "drifted process" for the time reversed representation. As we will see in the proof of Theorem 9, we will construct in the case $H > 1/2$ a drift $\widehat{u}^H_\cdot$ such that $\int_0^\cdot \widehat{u}_s ds = \mathcal{K}_H(g_\cdot)$, where g is a process. Actually in the case $H < 1/2$, the operator $\mathcal{K}_H$ does not map L^2 into a space of absolutely continuous functions (e.g. see (15) for the expressions of $\mathcal{K}_H$ when $H < 1/2$). So, although we can still write for $\overline{Y}^H$ a continuous extension formula from the Wiener case:

$$\overline{Y}_t^H = \overline{Y}_0^H + \widehat{U}_t^H + \widehat{B}_t^H,$$

the process $\widehat{U}^H$ is not in general of the form $\int_0^\cdot \widehat{u}_s^H ds$.

4.3 Proof of Theorem 9

In the sequel, we will use the letter X^H for a semi-martingale driven by the Bm W, and b^H to design its drift. The notation X^H means that the semi-martingale depends on H. We will have $X^{1/2} = Y^{1/2}$.

We need the following lemma:

Lemma 11. *Let X be a drifted Bm with drift $(b_t)_{t\in[0,T]}$ satisfying the assumptions of Theorem 8:*

$$X_t = x + \int_0^t b_s ds + W_t \ , \tag{28}$$

and let X its time reversed process:

$$\overline{X}_t = \overline{X}_0 + \int_0^t \widehat{b}_s ds + \widehat{W}_t.$$

Then for any $0 \leqslant t \leqslant T$, we have the following formula:

$$\int_0^{T-t} K_H(T-t,s) dX_s = -\int_{T-t}^{T} K_H(T-t, T-u) d\overline{X}_u.$$

Proof. We first prove the following equality

$$\int_\gamma^\delta K_H(T-t,s) dX_s = -\int_{T-\delta}^{T-\gamma} K_H(T-t, T-u) d\overline{X}_u. \tag{29}$$

where $0 < \gamma < \delta < T - t$. Remind that

$$K_H(T-t,s) = \frac{(T-t-s)^{H-1/2}}{\Gamma(H+1/2)} F\left(H-1/2, 1/2-H, H+1/2, 1-\frac{T-t}{s}\right)$$

where F denotes Gauss hypergeometric function (see *e.g.* (6; 15)). The function $z \mapsto F(H-1/2, 1/2-H, H+1/2, z)$ is holomorphic on the domain $\{z \in \mathbb{C}, z \neq 1, |arg(1-z)| < \pi\}$. It follows that the function

$$s \mapsto F\left(H-1/2, 1/2-H, H+1/2, 1-\frac{T-t}{s}\right)$$

is continuously differentiable on any interval $[\gamma, \delta]$, and so is the function $s \mapsto K_H(T-t,s)$. We deduce that $s \mapsto K_H(T-t,s)$ is C^1 on $[\gamma,\delta]$.

The integration by part formula w.r.t. the semimartingale X leads to

$$\int_\gamma^\delta K_H(T-t,s)dX_s = K_H(T-t,\delta)X_\delta - K_H(T-t,\gamma)X_\gamma - \int_\gamma^\delta K_H(T-t,s)X_s ds.$$

With the definition of $\overline{X}$ and the change of variable $u = T-t$, we then write:

$$\begin{aligned}\int_\gamma^\delta K_H(T-t,s)dX_s &= K_H(T-t,\delta)\overline{X}_{T-\delta} - K_H(T-t,\gamma)\overline{X}_{T-\gamma} \\ &\quad - \int_{T-\delta}^{T-\gamma} K_H(T-t,T-u)\overline{X}_u du.\end{aligned}$$

We deduce (29) by the integration by part formula w.r.t. the semimartingale $\overline{X}$.

To take the limit in (29) when (γ,δ) goes to $(0,T-t)$, we write thanks to Theorem 5:

$$\begin{aligned}&\int_\gamma^\delta K_H(T-t,s)b_s ds + \int_\gamma^\delta K_H(T-t,s)dW_s \\ &\quad = -\int_{T-\delta}^{T-\gamma} K_H(T-t,T-u)\widehat{b}_u du - \int_{T-\delta}^{T-\gamma} K_H(T-t,T-u)d\widehat{W}_u.\end{aligned}$$

Since W and $\widehat{W}$ are Brownian motion and $s \mapsto K_H(T-t,s) \in L^2(0,T)$, we can take the desired limit in the stochastic integrals. Moreover we also have $s \mapsto K_H(T-t,s) \in L^q(0,T)$ for $q \in (2, 2/(2H-1))$ and $\widehat{b} \in L^p(\Omega \times (0,T))$ for $p \in (1,2)$, which concludes the proof. □

Now we prove Theorem 9.

Proof. We divide the proof in two steps.

First step. Using the transfer principle and the isometry $\mathcal{K}_H$, it holds that

$$Y_t^H = y + \int_0^t K_H(t,s)dX_s^H$$

where

$$X_t^H = W_t + \int_0^t \mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)(r)dr.$$

Thanks to the condition **(i)**, we can apply Theorem 5 to the drifted Bm X^H with finite energy drift process $(b_t^H)_{t\in[0,T]}$ defined by

$$b_t^H = \mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)(t)\ .$$

If $(\widehat{\mathcal{F}}_t)_{t\in[0,T]}$ is the filtration generated by the reversed process $\overline{X}_t^H := X_{T-t}^H$, then there exists a $(\widehat{\mathcal{F}}_t)$-adapted processes $(\widehat{b}_t^H)_{t\in[0,T]} \in L^p(\Omega\times[0,T])$ for any $p\in(1,2)$, and a $(\widehat{\mathcal{F}}_t)$-Bm $(\widehat{W}_t^H)_{t\in[0,T]}$ such that

$$\widehat{W}_t^H = \overline{X}_t^H - \overline{X}_0^H - \int_0^t \widehat{b}_s^H ds.$$

We deduce from Lemma 11 that

$$\begin{aligned}\overline{Y}_t^H = &- \int_0^{T-t} s^{H-1/2}\left(\int_0^s (s-r)^{H-3/2}\widehat{b}_{T-r}^H r^{1/2-H} dr\right) ds \\ &- \int_t^T K_H(T-t,T-r)d\widehat{W}_r^H.\end{aligned}$$

We then write:

$$\overline{Y}_t^H - \overline{Y}_0^H = \int_0^t \widehat{u}_s^H ds + \widehat{B}_t^H,$$

where

$$\begin{aligned}\widehat{u}_s^H &= (T-s)^{H-1/2}\left(\int_0^{T-s} (T-s-r)^{H-3/2}\widehat{b}_{T-r}^H r^{1/2-H} dr\right) \\ \widehat{B}_t^H &= \int_0^T K_H(T,T-r)d\widehat{W}_r^H - \int_t^T K_H(T-t,T-r)d\widehat{W}_r^H.\end{aligned}$$

We compute the covariance of the centered Gaussian process $(\widehat{B}_t^H)_{t\in[0,T]}$:

$$\begin{aligned}E(\widehat{B}_s^H\widehat{B}_t^H) = &\left\langle K_H(T,T-\cdot), K_H(T,T-\cdot)\right\rangle_{L^2(0,T)} \\ &- \left\langle K_H(T,T-\cdot), K_H(T-s,T-\cdot)\mathbf{1}_{[s,T]}\right\rangle_{L^2(0,T)} \\ &- \left\langle K_H(T-t,T-\cdot)\mathbf{1}_{[t,T]}, K_H(T,T-\cdot)\right\rangle_{L^2(0,T)} \\ &+ \left\langle K_H(T-t,T-\cdot)\mathbf{1}_{[t,T]}, K_H(T-s,T-\cdot)\mathbf{1}_{[s,T]}\right\rangle_{L^2(0,T)}.\end{aligned}$$

Since

$$\begin{aligned}&\left\langle K_H(T-t,T-\cdot)\mathbf{1}_{[t,T]}, K_H(T-s,T-\cdot)\mathbf{1}_{[s,T]}\right\rangle_{L^2(0,T)} \\ &\qquad = \left\langle K_H(T-s,\cdot)\mathbf{1}_{[0,T-s]}, K_H(T-t,\cdot)\mathbf{1}_{[0,T-t]}\right\rangle_{L^2(0,T)} \\ &\qquad = \left\langle \mathbf{1}_{[0,T-t]}\ ,\ \mathbf{1}_{[0,T-s]}\right\rangle_{\mathcal{H}} \\ &\qquad = R_H(T-t,T-s),\end{aligned}$$

we deduce that

$$\begin{aligned}E(\widehat{B}_s^H\widehat{B}_t^H) &= R_H(T,T) - R_H(T-s,T) \\ &\quad - R_H(T,T-t) + R_H(T-s,T-t) \\ &= 1/2\Big(2T^{2H} - |T-s|^{2H} - T^{2H} + s^{2H} - T^{2H} - |T-t|^{2H} \\ &\quad + t^{2H} + |T-s|^{2H} + |T-t|^{2H} - |t-s|^{2H}\Big) \\ &= R_H(s,t)\ .\end{aligned}$$

Hence the process $(\widehat{B}_t^H)_{t\in[0,T]}$ is a fBm.

Remark moreover that writing the process $\widehat{u}_\cdot^H = \mathcal{O}_H(\widehat{b}_{T-\cdot})$ shows that it is continuous (see (10)).

Second step. Let us show that for all $t \in (0,T)$, $\int_0^t \widehat{u}_s^H ds \to \int_0^t \widehat{u}_s ds$ in $L^1(\Omega)$ when H tends to 1/2. We write

$$\int_0^t (\widehat{u}_s^H - \widehat{u}_s) ds = \int_0^t \mathcal{O}_H \left(\widehat{b}_{T-\cdot}^H - \widehat{u}_{T-\cdot}\right)(T-s) ds + \int_0^t \left(\mathcal{O}_H(\widehat{u}_{T-\cdot})(T-s) - \widehat{u}_s\right) ds \ . \tag{30}$$

First of all, we study the first term of the r.h.s. of (30). Lemma 1 implies that

$$\begin{aligned}\left|\int_0^t \mathcal{O}_H \left(\widehat{b}_{T-\cdot}^H - \widehat{u}_{T-\cdot}\right)(T-s) ds\right| &\leqslant \int_0^T \left|\mathcal{O}_H \left(\widehat{b}_{T-\cdot}^H - \widehat{u}_{T-\cdot}\right)(s)\right| ds \\ &\leqslant C(H_0) \int_0^T \left|\widehat{b}_{T-s}^H - \widehat{u}_{T-s}\right| s^{1/2-H_0} ds.\end{aligned}$$

We have thanks to Proposition 7:

$$\begin{aligned}\widehat{b}_{T-s}^H - \widehat{u}_{T-s} &= \lim_{h\downarrow 0} E\left[\frac{X_s^H - X_{s-h}^H - (X_s - X_{s-h})}{h} \Big| \widehat{\mathcal{F}}_{T-s}\right] \\ &= \lim_{h\downarrow 0} E\left[\frac{\int_{s-h}^s (b_r^H - u_r) dr}{h} \Big| \widehat{\mathcal{F}}_{T-s}\right] \\ &= E\left[b_s^H - u_s \Big| \widehat{\mathcal{F}}_{T-s}\right].\end{aligned}$$

So, by Jensen inequality and Fubini's theorem

$$E\left[\int_0^T s^{1/2-H_0} \left|\widehat{b}_{T-s}^H - \widehat{u}_{T-s}\right| ds\right] \leqslant E\left[\int_0^T s^{1/2-H_0} \left|b_s^H - u_s\right| ds\right]$$

and then

$$E\left|\int_0^t \mathcal{O}_H \left(\widehat{b}_{T-\cdot}^H - \widehat{u}_{T-\cdot}\right)(T-s) ds\right| \leqslant \int_0^T s^{1/2-H_0} E\left|b_s^H - u_s\right| ds. \tag{31}$$

We have

$$b_t^H - u_t = \mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)(t) - u_t$$

and since

$$\mathcal{K}_H^{-1}\left(\int_0^\cdot u_s^H ds\right)(t) = t^{H-1/2} D_{0+}^{H-1/2}(s^{1/2-H} u_s^H)(t) \ ,$$

we get

$$\begin{aligned}\left|b_t^H - u_t\right| &\leqslant \left|\frac{t^{1/2-H} u_t^H}{\Gamma(3/2-H)} - u_t\right| \\ &\qquad + \frac{t^{1/2-H}(H-1/2)}{\Gamma(3/2-H)} \left|\int_0^t \frac{t^{1/2-H} u_t^H - s^{1/2-H} u_s^H}{(t-s)^{H+1/2}} ds\right| \\ &\leqslant \frac{t^{1/2-H}}{\Gamma(3/2-H)} |u_t^H - u_t| + \left(\frac{t^{1/2-H}}{\Gamma(3/2-H)} - 1\right) |u_t| \\ &\qquad + \frac{t^{1/2-H}(H-1/2)}{\Gamma(3/2-H)} \big(I_1(H,t) + I_2(H,t)\big)\end{aligned} \tag{32}$$

where

$$I_1(H,t) = \int_0^t \frac{t^{1/2-H}|u_t^H - u_s^H|}{(t-s)^{H+1/2}} ds,$$
$$I_2(H,t) = \int_0^t \frac{|u_s^H|(s^{1/2-H} - t^{1/2-H})}{(t-s)^{H+1/2}} ds.$$

Using $E|u_t^H - u_s^H| \leqslant C|t-s|^\eta$ with $\eta > H - \frac{1}{2}$ implies that $E\big(I_1(H,t)\big)$ is bounded uniformly for $(H,t) \in (1/2, H_0) \times (0,T)$. From the inequality $|t^{1/2-H} - s^{1/2-H}| \leqslant (H-1/2)|t-s|(t/2)^{-1/2-H}$ for $t \geqslant s \geqslant t/2$ we deduce that

$$E\big(I_2(H,t)\big) \leqslant c\int_{\frac{t}{2}}^t (t-s)^{1/2-H} ds + c\int_0^{\frac{t}{2}} (s^{1/2-H} - t^{1/2-H})(t-s)^{1/2-H} ds \tag{33}$$

and $E\big(I_2(H,t)\big)$ is also bounded uniformly in (H,t). Now, since $E|u_t^H - u_t|$ tends to 0 as $H \downarrow 1/2$ we have

$$E\left|b_t^H - u_t\right| \longrightarrow 0 \quad \text{as } H \downarrow \tfrac{1}{2} \text{ for almost all } t. \tag{34}$$

Let $1 < p < \frac{1}{H_0} < 2$, we use Hölder inequality

$$\int_0^T s^{p(1/2-H_0)} E|b_s^H - u_s|^p ds \leqslant \left(\int_0^T s^{\frac{2p(\frac{1}{2}-H_0)}{2-p}} ds\right)^{\frac{2-p}{2}} \left(E\int_0^T |b_s^H - u_s|^2 ds\right)^{\frac{p}{2}}.$$

Thanks to the hypothesis **(ii)**, $\{s \mapsto s^{1/2-H_0} E|b_s^H - u_s| \ ; \ H \in [1/2, H_0]\}$ is bounded in $L^p\big([0,T], \frac{dt}{T}\big)$, thus this family is uniformly integrable.

By (34),

$$s^{1/2-H_0} E|b_s^H - u_s| \to 0 \quad , \quad \tfrac{ds}{T} \ \ a.s.$$

when H tends to $1/2$, so this convergence also holds in $L^1\left([0,T], \frac{dt}{T}\right)$. Reporting this convergence result in (31),

$$E\left|\int_0^t \mathcal{O}_H(\widehat{b}_{T-\cdot}^H - \widehat{u}_{T-\cdot})(T-s) ds\right| \longrightarrow 0$$

when H tends to $1/2$.

We now study the second term of the r.h.s. of (30). We write:

$$\begin{aligned}\int_0^t \mathcal{O}_H(\widehat{u}_{T-\cdot})(T-s) ds &= \int_{T-t}^T \mathcal{O}_H(\widehat{u}_{T-\cdot})(r) dr \\ &= \mathcal{K}_H(\widehat{u}_{T-\cdot})(T) - \mathcal{K}_H(\widehat{u}_{T-\cdot})(T-t).\end{aligned}$$

By Theorem 5, $\widehat{u} \in L^p(0,T)$ a.s. for any $p \in (1,2)$, and consequently $\widehat{u}$ satisfies a.s. the hypothesis of Lemma 1 which then yields the following estimation and convergence:

$$\begin{aligned}\left|\int_0^t \mathcal{O}_H\left(\widehat{u}_{T-\cdot}\right)(T-s) ds\right| &\leqslant \int_0^T \left|\mathcal{O}_H\left(\widehat{u}_{T-\cdot}\right)(s)\right| ds \\ &\leqslant C(H_0)\int_0^T |\widehat{u}_{T-s}| \left(s^{1/2-H_0} \vee 1\right) ds, \\ \lim_{H\downarrow 1/2} \int_0^t \mathcal{O}_H(\widehat{u}_{T-\cdot})(T-s) ds &= \int_0^T \widehat{u}_{T-r} dr - \int_0^{T-t} \widehat{u}_{T-r} dr \\ &= \int_0^t \widehat{u}_s ds \quad a.s.\end{aligned}$$

Since for any $p \in (1,2)$, $\widehat{u} \in L^p(\Omega \times (0,T))$, we have

$$\int_0^T |\widehat{u}_{T-s}|\, (s^{1/2-H_0} \vee 1) ds \in L^1(\Omega),$$

and we can apply the dominated convergence theorem and write

$$\lim_{H \downarrow 1/2} E \left| \int_0^t \mathcal{O}_H(\widehat{u}_{T-\cdot})(T-s) ds - \int_0^t \widehat{u}_s ds \right| = 0.$$

So we conclude that for all $t \in (0,T)$, $\int_0^t \widehat{u}^H_s ds \to \int_0^t \widehat{u}_s ds$ in $L^1(\Omega)$ when H tends to $1/2$.
Using Lemma 3.2 in (5), we have

$$E|B^H_t - W_t|^2 \leqslant (cT)^2 |H-1/2|^2$$

and then B^H_t tends to W_t in $L^2(\Omega)$ when H tends to $1/2$.
Since $\overline{Y}^H_t = Y^H_{T-t} = y + \int_0^{T-t} u^H_s ds + B^H_{T-t}$ we deduce that $\overline{Y}^H_t$ tends to $\overline{Y}^{1/2}_t$ in $L^1(\Omega)$ when H tends to $1/2$. Hence, for all $t \in (0,T)$

$$\lim_{H \downarrow 1/2} \widehat{B}^H_t = \widehat{W}_t \quad \text{ in } L^1(\Omega),$$

which concludes the proof of the theorem. $\square$

5 Application to stochastic differential equations driven by a fBm

First of all, we apply our result to the reversal of a fBm. We yet consider that B^H is a fBm having the integral representation $B^H_t = \int_0^t K_H(t,s) dW_s$. It is well known (see (17)) that the reversed process $\overline{W}$ solves:

$$d\overline{W}_t = -\frac{\overline{W}_t}{T-t} dt + d\widehat{W}_t,$$

where the Brownian motion $\widehat{W}_t$ is given by

$$\widehat{W}_t = W_{T-t} - W_T + \int_{T-t}^T \frac{W_s}{s} ds.$$

Therefore, thanks to Theorem 9, we deduce that the reversed fBm reads:

$$\overline{B}^H_t = \overline{B}^H_0 + \int_0^t (T-s)^{H-1/2} \int_0^{T-s} r^{-1/2-H} (T-s-r)^{H-3/2} W_r dr ds + \widehat{B}^H_t$$

where the fBm $(\widehat{B}^H_t)_{t \in [0,T]}$ is given by

$$\widehat{B}^H_t = \int_0^T K_H(T, T-u) d\widehat{W}_u - \int_t^T K_H(T-t, T-u) d\widehat{W}_u.$$

This situation can be extended in the case of stochastic differential equations driven by B^H.

5.1 SDE driven by a single fBm

Using successively Theorem 9, Theorem 8 and the results in (17), we can state the following proposition:

Proposition 12. **(a)** *Let Y^H be the process defined as the unique solution of*

$$Y_t^H = y + \int_0^t u(Y_s^H)ds + B_t^H \quad , \quad 0 \leqslant t \leqslant T \ , \tag{35}$$

where the function u is bounded with bounded first derivative. Then there exists a family of processes $\left(\widehat{u}^H\right)_{H \geqslant 1/2}$ and a family of fBm $(\widehat{B}^H)_{H \geqslant 1/2}$ such that the time reversed process $(\overline{Y}_t^H)_{t \in [0,T]}$ defined by $\overline{Y}_t^H = Y_{T-t}^H$ satisfies

$$\overline{Y}_t^H = \overline{Y}_0^H + \int_0^t \widehat{u}_s^H ds + \widehat{B}_t^H,$$

with the following $L^1(\Omega)$ convergences

$$\lim_{H \downarrow 1/2} \widehat{B}_t^H = B_T^{1/2} - B_{T-t}^{1/2} + \int_{T-t}^T \partial_x \log p_s(Y_s^{1/2}) \ ds = \widehat{B}_t^{1/2}, \tag{36}$$

$$\lim_{H \downarrow 1/2} \int_0^t \widehat{u}_s^H ds = \int_0^t \left(-u(\overline{Y}_s^{1/2}) + \partial_x \log p_{T-s}(\overline{Y}_s^{1/2}) \right) ds \tag{37}$$

for all $t \in (0,T)$, where $(t,y) \mapsto p_t(y)$ is the density of the law of $Y_t^{1/2}$ solution of

$$Y_t^{1/2} = y + \int_0^t u(Y_s^{1/2})ds + B_t^{1/2}.$$

(b) *The process $\overline{Y}^H$ is not a "fractional diffusion", i.e. of the form (12).*

Proof. It is proved in (15) that there exists a unique strong solution of stochastic differential equation (35).

First step. In order to prove **(a)**, we have to verify that all the assumptions of Theorem 9 are fulfilled. We recall that

$$b_t^H = \mathcal{K}_H^{-1} \left(\int_0^{\cdot} u(Y_s^H) ds \right)(t).$$

The process $(b_t^H)_{t \in [0,t]}$ satisfies the Novikov condition (16) as noticed in the section 3.3 p.110−111 in (15) and the condition **(i)** holds true.

We check the assumptions of **(ii)**.

The trajectories of the process $(Y_t)_{t \in [0,T]}$ are Hölder continuous of order $H - \epsilon$ for all $\epsilon > 0$ (see (16; 15)). Since the function u has a bounded first derivative, the process $(u(Y_t))_{t \in [0,T]}$ has

Hölder continuous trajectories of order $H-1/2+\epsilon$ for all $1/2>\epsilon>0$. In order to check that the condition **a)** of **(ii)** is fulfilled, it remains to write that for $s\leqslant t$

$$
\begin{aligned}
E|u(Y_t^H)-u(Y_s^H)| &\leqslant \|u'\|_\infty E|Y_t^H-Y_s^H| \\
&\leqslant \|u'\|_\infty \left(\int_s^t E|u(Y_r^H)|dr + E|B_t^H-B_s^H|\right) \\
&\leqslant \|u'\|_\infty \left(\|u\|_\infty |t-s| + |t-s|^H\right) \ .
\end{aligned}
$$

The convergence of $u(Y_t^H)$ toward $u(Y_t^{\frac{1}{2}})$ in $L^1(\Omega)$ will be a consequence of the convergence of $Y_t^H \to Y_t^{\frac{1}{2}}$. By the Gronwall Lemma, we have

$$
|Y_t^H - Y_t^{\frac{1}{2}}| \leqslant |B_t^H - B_t^{1/2}| + c\int_0^t |B_s^H - B_s^{1/2}| \exp\big(c(t-s)\big) ds,
$$

and using $\lim_{H\downarrow 1/2} B_t^H = B_t^{1/2}$ in $L^2(\Omega)$ and $\sup_{H\geq 1/2} E|B_t^H - B_t^{1/2}|^2 \leq c$ implies that for almost all $t\in[0,T]$,

$$
\lim_{H\downarrow 1/2} Y_t^H = Y_t^{1/2} \quad \text{in } L^2(\Omega)\ .
$$

Then **c)** of **(ii)** is true.

Using analogues estimates that those carried out in (32), we get that

$$
\begin{aligned}
&\Gamma(3/2-H)\ |b_t^H| \\
&\quad\leqslant |t^{1/2-H}u(Y_t^H)| + t^{1/2-H}(H-1/2)\left\{\int_0^t \frac{t^{1/2-H}|u(Y_t^H)-u(Y_s^H)|}{(t-s)^{H+1/2}}ds\right. \\
&\qquad\qquad\left. + \int_0^t \frac{|u(Y_s^H)|(s^{1/2-H}-t^{1/2-H})}{(t-s)^{H+1/2}}ds\right\} \\
&\quad\leqslant \|u\|_\infty t^{1/2-H} + t^{1/2-H}(H-1/2)\left\{\|u'\|_\infty \int_0^t \frac{t^{1/2-H}|Y_t^H-Y_s^H|}{(t-s)^{H+1/2}}ds\right. \\
&\qquad\qquad\left. + \|u\|_\infty \int_0^t \frac{(s^{1/2-H}-t^{1/2-H})}{(t-s)^{H+1/2}}ds\right\}.
\end{aligned}
\tag{38}
$$

Arguing as in (33) we get that the last term of the right hand side of (38) is bounded when H varies. It is easy to see that $E|Y_t^H - Y_s^H|^{2p} \leqslant c_p|t-s|^{2pH}$ for any $p\geq 1$. Moreover, Lemma 2 yields that there exists a square integrable random variable $\xi_{H,\epsilon}$ such that $|Y_t^H - Y_s^H| \leqslant \xi_{H,\epsilon}|t-s|^{H-\epsilon}$ for any $0<\epsilon<H$ and $\sup_{H\in[1/2,1)} E\xi_{H,\epsilon}^2 \leqslant C_\epsilon$. Then we deduce that

$$
\sup_{H\in[1/2,1)} E\int_0^T |b_t^H|^2 dt \leqslant C\ ,
$$

and condition **b)** of **(ii)** holds.

By (7), the expression (21) has a particular form in the case of diffusion processes: actually the reversed drift is $(-u(\overline{Y}_t^{1/2}) + \partial_x \ln p_{T-t}(\overline{Y}_t^{1/2}))_{t\in[0,T]}$. The proof of **(a)** is then completed.

Second step. We now prove **(b)**. We have to use Theorem 8 to obtain the explicit form of the drift. In order to verify the conditions 1, 2 and 3 of Theorem 8, we compute the Malliavin derivatives with respect to the process $(X_t)_{t\in[0,T]}$ defined by

$$X_t = W_t + \int_0^t \mathcal{K}_H^{-1}\left(\int_0^{\cdot} u(Y_s)ds\right)(r)dr$$

which is a Bm under the probability measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = G$ where G is given by (20). Let $Y_t = \int_0^t K_H(t,s)dX_s$ where we omit the index H for Y for simplicity. In view of the form of Y, $(Y_t)_{t\in[0,T]}$ is a fBm with respect to the new probability measure $\mathbb{Q}$ and we have the following relations between the Malliavin derivative with respect to X (denoted by D) and the Malliavin derivative with respect to Y (denoted D^Y): for any random variable $F \in \mathbb{D}^{1,2}$

$$\mathcal{K}_H^*\big(D^Y F\big) = DF\ .$$

Let $\alpha = H - 1/2$, using (9) one writes

$$\begin{aligned}
b_t &= t^{\alpha} D_{0+}^{\alpha}\left(s^{-\alpha}u(Y_s)\right)(t)\\
&= \frac{1}{\Gamma(1-\alpha)}\left(t^{-\alpha}u(Y_t) + \alpha t^{\alpha}\int_0^t \frac{t^{-\alpha}u(Y_t) - s^{-\alpha}u(Y_s)}{(t-s)^{\alpha+1}}ds\right),
\end{aligned}$$

and one remarks that for any $r \leqslant t$

$$\begin{aligned}
D_r^Y b_t = \frac{1}{\Gamma(1-\alpha)}\Big(&t^{-\alpha}u'(Y_t)\mathbf{1}_{r\leqslant t}\\
&+ \alpha t^{\alpha}\int_0^t \frac{t^{-\alpha}u'(Y_t)\mathbf{1}_{r\leqslant t} - s^{-\alpha}u'(Y_s)\mathbf{1}_{r\leqslant s}}{(t-s)^{\alpha+1}}ds\Big).
\end{aligned}$$

The following computations are quite the same one that those carried out in the proof of Lemma 14 of (11) in a different framework. For sake of completeness, we include them.

$$\begin{aligned}
D_r b_t &= \frac{1}{\Gamma(1-\alpha)}\Big(t^{-\alpha}u'(Y_t)\mathcal{K}_H^*\big(\mathbf{1}_{[0,t]}\big)(r)\\
&\quad + \alpha t^{\alpha}\int_0^t \frac{t^{-\alpha}u'(Y_t)\mathcal{K}_H^*\big(\mathbf{1}_{[0,t]}\big)(r) - s^{-\alpha}u'(Y_s)\mathcal{K}_H^*\big(\mathbf{1}_{[0,s]}\big)(r)}{(t-s)^{\alpha+1}}ds\Big)\\
&= \frac{1}{\Gamma(1-\alpha)}\Big(t^{-\alpha}u'(Y_t)K_H(t,r)\mathbf{1}_{r\leqslant t}\\
&\quad + \alpha t^{\alpha}\int_0^t \frac{t^{-\alpha}u'(Y_t)K_H(t,r)\mathbf{1}_{r\leqslant t} - s^{-\alpha}u'(Y_s)K_H(s,r)\mathbf{1}_{r\leqslant s}}{(t-s)^{\alpha+1}}ds\Big)\\
&:= f_1(t,r) + f_2(t,r) + f_3(t,r) + f_4(t,r),
\end{aligned}$$

where

$$\begin{aligned}
f_1(t,r) &= \frac{1}{\Gamma(1-\alpha)}u'(Y_t)K_H(t,r)(t-r)^{-\alpha}\\
f_2(t,r) &= \frac{\alpha u'(Y_t)}{\Gamma(1-\alpha)}\int_r^t \frac{K_H(t,r) - K_H(s,r)}{(t-s)^{\alpha+1}}ds\\
f_3(t,r) &= \frac{\alpha}{\Gamma(1-\alpha)}\int_r^t \frac{u'(Y_t) - u'(Y_s)}{(t-s)^{\alpha+1}}K_H(s,r)ds\\
f_4(t,r) &= \frac{\alpha t^{\alpha}}{\Gamma(1-\alpha)}\int_r^t \frac{t^{-\alpha} - s^{-\alpha}}{(t-s)^{\alpha+1}}u'(Y_s)K_H(s,r)ds
\end{aligned}$$

for $r \leqslant t$ and the functions f_i, $i = 1, ..., 4$ vanish when $r > t$. Remind that (see (3)) that

$$K_H(t,r) = c_H\big(H - 1/2\big)r^{-\alpha}\int_r^t (\theta - r)^{\alpha-1}\theta^\alpha d\theta \ , \tag{39}$$

we have

$$\begin{aligned}
f_2(t,r) &= \frac{c_H\alpha^2}{\Gamma(1-\alpha)}u'(Y_t)\int_r^t \frac{r^{-\alpha}\int_s^t(\theta-r)^{\alpha-1}\theta^\alpha d\theta}{(t-s)^{\alpha+1}}ds \\
&= \frac{c_H\alpha^2}{\Gamma(1-\alpha)}u'(Y_t)r^{-\alpha}\int_r^t\int_r^\theta (t-s)^{-\alpha-1}ds(\theta-r)^{\alpha-1}\theta^\alpha d\theta \\
&= f_5(t,r) - f_1(t,r) \quad \text{with} \\
f_5(t,r) &= \frac{c_H\alpha}{\Gamma(1-\alpha)}u'(Y_t)r^{-\alpha}\int_r^t (t-\theta)^{-\alpha}(\theta-r)^{\alpha-1}\theta^\alpha d\theta,
\end{aligned}$$

and it follows that

$$D_r b_t = f_3(t,r) + f_4(t,r) + f_5(t,r). \tag{40}$$

The above expression implies that the process $(D_r b_t)_{t\in[0,T]}$ is adapted with respect to the filtration generated by the Bm X because it is the same one that the filtration generated by the process Y. Therefore if we have

$$E_{\mathbb{Q}}\int_0^T\int_0^T |D_r b_t|^2 dr dt < \infty \ , \tag{41}$$

the assumptions 1, 2 and 3 of Theorem 8 will be checked.

Using the fact that $(Y_t)_{t\in[0,T]}$ is a fBm under the probability $\mathbb{Q}$ and Lemma 2, we get that for any $\epsilon > 0$, there exists a square integrable random variable $\zeta_{H,\epsilon}$ such that

$$|Y_t - Y_s| \leqslant \zeta_{H,\epsilon}|t-s|^{H-\epsilon}.$$

Since the function u is Lipschitz, we get for $0 < \epsilon < 1/2$

$$E_{\mathbb{Q}}\int_0^T\int_0^T |f_3(t,r)|^2 dr dt \leqslant E_{\mathbb{Q}}\int_0^T\int_0^T c\ \zeta_{H,\epsilon}^2\left|\int_r^t (t-s)^{-\frac{1}{2}-\epsilon}K_H(s,r)ds\right|^2 dr dt \ .$$

Reporting in the expression (39) the fact that $\theta^\alpha \leqslant r^\alpha$ for $\theta \geqslant r$ yields

$$|K_H(s,r)| \leqslant c_H(s-r)^\alpha. \tag{42}$$

We conclude that

$$E_{\mathbb{Q}}\int_0^T\int_0^T |f_3(t,r)|^2 dr dt \leqslant c < \infty. \tag{43}$$

From the inequalities (42) and $|t^{-\alpha} - s^{-\alpha}| \leqslant \alpha(t-s)t^{-\alpha+1}$ for $t \geqslant s \geqslant r$, we get

$$|f_4(t,r)| \leqslant \frac{\alpha c_H\|u'\|_\infty}{\Gamma(1-\alpha)}\ t^\alpha\int_r^t \frac{t^{-\alpha-1}}{(t-s)^\alpha}(s-r)^\alpha ds \leqslant c\ t^{-1}\int_r^t\left(\frac{s-r}{t-s}\right)^\alpha ds \ ,$$

and the change of variable $s = (t-r)\xi + r$ yields

$$|f_4(t,r)| \leqslant c\, t^{-1}(t-r) \int_0^1 \xi^\alpha (1-\xi)^{-\alpha} d\xi \leqslant c\, \beta(1-\alpha, \alpha+1)\ .$$

We finally get

$$E_{\mathbb{Q}} \int_0^T \int_0^T |f_4(t,r)|^2 dr dt < \infty. \tag{44}$$

We use another time the inequality $\theta^\alpha \leqslant r^\alpha$ for $\theta \geqslant r$ and the change of variable $\theta = (t-r)\xi + r$ in order to have

$$|f_5(t,r)| \leqslant \frac{\alpha c_H \|u'\|_\infty}{\Gamma(1-\alpha)} (t-r)^{1-\alpha} \beta(1-\alpha, \alpha)\ ,$$

and consequently

$$E_{\mathbb{Q}} \int_0^T \int_0^T |f_5(t,r)|^2 dr dt < \infty. \tag{45}$$

The expression (40) and the inequalities (43), (44) and (45) imply that (41) is satisfied. Consequently, Theorem 8 asserts that there exists a reversed drift $\widehat{b}$ of the form (21) for the time reversal of X. Since the drift $\widehat{u}^H$ of $\overline{Y}^H$ reads $\widehat{u}^H_\cdot = \mathcal{O}_H(\widehat{b}_{T-\cdot})$, we deduce that $\overline{Y}^H$ cannot be a "fractional diffusion". $\square$

5.2 A remark on fractional SDE with a non linear diffusion coefficient

We are now interested in fractional SDE with a non linear diffusion coefficient. Let X^H be the solution of

$$X_t^H = x_0 + \int_0^t \sigma(X_s^H) dB_s^H + \int_0^t b(X_s^H)\, ds, \quad t \in [0,T], \tag{46}$$

where the stochastic integral is understood in the Young sense.

Let us assume the conditions given in (10) to ensure that the time reversed process of the diffusion $X^{1/2}$ is again a diffusion:

1. $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions satisfying the hypothesis: there exists a constant $K > 0$ such that for every $x, y \in \mathbb{R}$ we have

$$\begin{aligned} &|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leqslant K\, |x-y|\,, \\ &|\sigma(x)| + |b(x)| \leqslant K(1+|x|). \end{aligned}$$

2. For any $t \in (0,T)$, $X_t^{1/2}$ has a density p_t.

3. For any $t_0 \in (0,T)$, for any bounded open set $O \subset \mathbb{R}$,

$$\int_{t_0}^T \int_O \left|\partial_x(\sigma^2(x) p_t(x))\right| dx dt < +\infty.$$

Moreover, assume that $|\sigma| \geq c > 0$ does not vanish. As in (15) we set $Y_t^H = h(X_t^H)$ where $h(x) = \int_0^x \frac{dy}{\sigma(y)}$. Using the change of variables formula, we obtain that Y verifies

$$Y_t^H = y_0 + B_t^H + \int_0^t \frac{b(h^{-1}(Y_s^H))}{\sigma(h^{-1}(Y_s^H))} ds, \quad t \in [0,T].$$

If b and σ are such that $b \circ h^{-1}/\sigma \circ h^{-1}$ is bounded with bounded first derivative, we can apply our previous theorem and obtain a time reversed representation for $\overline{Y}_t^H$ (which is continuous in $L^1(\Omega)$ when $H \downarrow 1/2$):

$$\overline{Y}_t^H = \overline{Y}_0^H + \int_0^t \widehat{u}_s^H ds + \widehat{B}_t^H.$$

But $\overline{X}_t^H = h^{-1}(\overline{Y}_t^H)$, so:

$$\overline{X}_t^H = \overline{X}_0^H + \int_0^t \sigma(\overline{X}_s^H)\widehat{u}_s^H ds + \int_0^t \sigma(\overline{X}_s^H) d\widehat{B}_s^H.$$

If we assume that σ is bounded, the derivative of h^{-1} will also be bounded, hence $\overline{X}_t^H$ is continuous in $L^1(\Omega)$ when $H \downarrow 1/2$. Those of $\int_0^t \sigma(\overline{X}_s^H)\widehat{u}_s^H ds$ is ensured by $\widehat{u}^H \in L^p(\Omega \times [0,T])$ for $p \in (1,2)$ and σ Lipschitz. As a consequence, the stochastic integral $\int_0^t \sigma(\overline{X}_s^H) d\widehat{B}_s^H$ is also continuous in $L^1(\Omega)$ when $H \downarrow 1/2$.

6 H-deformation of Nelson derivatives

In this section we explore one possible way to construct dynamical operators acting on diffusions driven by a fractional Brownian motion.

For $1/2 \leqslant H < 1$ we denote by Υ_y^H the vector space of all processes Y^H of the form

$$Y_t^H = y + \int_0^t u_s^H ds + \sigma B_t^H ,$$

where $\sigma \in \mathbb{R}$ and $(u_t^H)_{t\in[0,T]}$ is a $\mathcal{F}_t$-adapted squared integrable process such that the process $b^H := \mathcal{K}_H^{-1}\left(\int_0^{\cdot} u_s^H ds\right)$ satisfies the Novikov condition (16) and the conditions 1, 2 and 3 of Theorem 8,

In particular, $\Upsilon_0^{1/2}$ is the space of all drifted Bm starting from 0 with a constant diffusion coefficient and a $\mathcal{F}_t$-adapted square integrable drift satisfying (16).

The following map is then well defined:

$$T_H : \left\{ \begin{array}{ccc} \Upsilon_0^{1/2} & \longrightarrow & \Upsilon_y^H \\ X & \longmapsto & y + \int_0^{\cdot} K_H(\cdot,s) dX_s \end{array} \right.$$

Let $Y^H \in \Upsilon_y^H$ be of the form $Y_t^H = y + \int_0^t u_s^H ds + \sigma B_t^H$, then clearly $T_H(X) = Y$ with $X_t = \int_0^t \mathcal{K}_H^{-1}(\int_0^{\cdot} u_s ds)(r) dr + \sigma W_t$, so T_H is surjective. Assume moreover that $T_H(X) = 0$ with $X_t = \int_0^t b_s ds + \sigma W_t$. Thus $T_H(X)_t = y + \int_0^t u_s ds + \sigma B_t^H = 0$ with $u(s) = \mathcal{K}_H(b)(s)$. Since B^H

is not absolutely continuous, this implies that $\sigma = 0$. By differentiating, we obtain $u = 0$. So $X = 0$ and T_H is one to one. Consequently, the application T_H is an isomorphism.

In the case $H = 1/2$, the drift of an element of $\Upsilon_W^{1/2}$ has a dynamical meaning in the sense of Nelson (*c.f.* Proposition 7). The isomorphism T_H provides us a way to introduce an equivalent notion in the general case $H \in (1/2, 1)$.

Definition 13. *Let $Y \in \Upsilon_{B^H}^H$ with $H \in (1/2, 1)$. The quantities*

$$\begin{aligned}
\mathbf{D}_+^{\mathbf{H}} Y_t &= \mathcal{O}_H \circ \mathbf{D}_+ \circ T_H^{-1}(Y)_t, \\
\mathbf{D}_-^{\mathbf{H}} Y_t &= \mathcal{O}_H \circ \mathbf{D}_- \circ T_H^{-1}(Y)_t
\end{aligned}$$

are well defined and respectively called the H-forward Nelson and H-backward Nelson derivative of Y at time t.

The fact that these operators are well defined is a consequence of Theorem 9, and moreover we have for all $t \in (0, T)$,

$$u_t = \mathbf{D}_+^{\mathbf{H}} Y_t.$$

So we have constructed an operator which associates to a process $Y \in \Upsilon_{B^H}^H$ its drift. However this operator does not calculate a difference rate directly on the process Y involved, but via a transfer principle. Moreover the map T_H hides difficulties as regards explicit calculation.

A Appendix

A.1 Proof of Theorem 5

Proof. Set $d\mathbb{Q} = G \cdot d\mathbb{P}$ where

$$G = \exp\left(-\int_0^T b_s dW_s - 1/2 \int_0^T b_s^2 ds \right) .$$

Since b fulfills the Novikov condition, the Girsanov theorem shows that $(X_t)_{t \in [0,T]}$ is a $(\mathcal{F}_t, \mathbb{Q})$-Bm.

Lemma 14. *Let $(\widehat{\mathcal{F}}_t)$ be the natural filtration generated by $\overline{X}$. Then the process $(W_t^{(1)})_{t \in [0,T]}$ defined by*

$$W_t^{(1)} := \overline{X}_t - \overline{X}_0 - \int_0^t \frac{\overline{X}_r}{T - r} dr$$

is a $(\widehat{\mathcal{F}}_t, \mathbb{Q})$-Bm.

Proof. We have

$$W_t^{(1)} = X_{T-t} - X_T + \int_0^t \frac{X_{T-r}}{T - r} dr.$$

We write $\widehat{\mathcal{F}}_s = \sigma(X_{T-s}) \vee \mathcal{G}_{T-s}$ where $\mathcal{G}_{T-s} = \sigma(X_r - X_{r'}, T - s \leqslant r < r' \leqslant T)$. For $s < t$, we have:

$$E_{\mathbb{Q}} \left[W_t^{(1)} - W_s^{(1)} \middle| \widehat{\mathcal{F}}_s \right] = E_{\mathbb{Q}} \left[X_{T-t} - X_{T-s} \middle| \widehat{\mathcal{F}}_s \right] + \int_s^t \frac{E_{\mathbb{Q}}[X_{T-r} | \widehat{\mathcal{F}}_s]}{T - r} dr.$$

But $X_{T-t} - X_{T-s}$ is independent of $\mathcal{G}_{T-s}$ and for all $r \in [s,t]$, $X_{T-r} = X_{T-r} - X_0$ is also independent of $\mathcal{G}_{T-s}$. Therefore:

$$E_{\mathbb{Q}}\left[W_t^{(1)} - W_s^{(1)} \left| \widehat{\mathcal{F}}_s \right.\right] = E_{\mathbb{Q}}\left[X_{T-t} - X_{T-s} \left| X_{T-s} \right.\right] + \int_s^t \frac{E_{\mathbb{Q}}\left[X_{T-r} \left| X_{T-s} \right.\right]}{T-r} dr.$$

Since $(X_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-Bm and $E_{\mathbb{Q}}\left[(X_{T-u} - \alpha(u)X_{T-s})X_{T-s}\right] = 0$, we can write

$$E_{\mathbb{Q}}\left[X_{T-u} - X_{T-s} \left| X_{T-s} \right.\right] = \alpha(u) X_{T-s} = \frac{T-u}{T-s} X_{T-s}$$

Thus

$$E_{\mathbb{Q}}\left[W_t^{(1)} - W_s^{(1)} \left| \widehat{\mathcal{F}}_s \right.\right] = \left[\frac{T-t}{T-s} - 1 + \int_s^t \frac{dr}{T-s}\right] X_{T-s} = 0.$$

and $W^{(1)}$ is a $(\widehat{\mathcal{F}}_t, \mathbb{Q})$-martingale. The fact that the quadratic variation of the continuous $(\widehat{\mathcal{F}}_t, \mathbb{Q})$-martingale $W^{(1)}$ is equal to t together with Lévy theorem conclude the proof of the lemma. □

Since $\mathbb{P} \sim \mathbb{Q}$, Girsanov theorem insures the existence of a $(\widehat{\mathcal{F}}_t)$-adapted process $(a_t)_{t\in[0,T]}$ such that

$$\widehat{W}_t := W_t^{(1)} - \int_0^t a_s ds$$

is a $(\widehat{\mathcal{F}}_t, \mathbb{P})$-Bm. The process $(a_t)_{t\in[0,T]}$ has finite energy with respect to $\mathbb{P}$ which has finite entropy with respect to $\mathbb{Q}$ (see Lemma 3.1 in (7)).
We then write

$$\widehat{W}_t = \overline{X}_t - \overline{X}_0 - \int_0^t \widehat{b}_s ds$$

where

$$\widehat{b}_s = a_s + \frac{\overline{X}_s}{T-s} \tag{47}$$

is a $(\widehat{\mathcal{F}}_s)$-adapted process with *a priori* only local finite energy, namely finite energy on any time interval $[0,\tau]$ for $\tau < T$.
However, one can prove that $\widehat{b} \in L^p(\Omega \times [0,T])$, $1 < p < 2$. The expression (47) indeed gives:

$$\begin{aligned} |\widehat{b}_{T-s}| &\leqslant |a_{T-s}| + \frac{1}{s}\left(|W_s| + \int_0^s |b_r| dr\right) \\ &\leqslant |a_{T-s}| + \frac{1}{s}\left(|W_s| + \sqrt{s}\sqrt{\int_0^s |b_r|^2 dr}\right). \end{aligned}$$

For any $p \in (1,2)$, $\frac{W_s}{s} \in L^p(\Omega \times [0,T])$. Finally, with $\int_0^T s^{-p/2} ds < \infty$, $b \in L^2(\Omega \times [0,T])$ and the Jensen inequality applied with the convex function $x \mapsto x^{2/p}$, one can deduce that $\widehat{b} \in L^p(\Omega \times [0,T])$. □

A.2 Proof of Theorem 8

Proof. We follow the ideas of Föllmer.

We denote $\mathcal{G}_t = \widehat{\mathcal{F}}_{T-t} = \sigma\{X_u \,;\, t \leqslant u \leq T\} = \mathcal{F}_{[t,T]} \vee \sigma\{X_t\}$, where $\mathcal{F}_{[t,T]}$ denotes the sigma-field generated by the increments of the Bm X between t and T.

Remind that the drift $(b_t)_{t\in[0,T]}$ can be expressed in term of forward Nelson derivative. The reversed drift is expressed thanks to (backward) Nelson derivative:

$$\hat{b}_{T-t} = -\lim_{h\downarrow 0} E\left(\frac{X_t - X_{t-h}}{h}\middle| \mathcal{G}_t\right) \qquad \text{in } L^2(\Omega)\ . \tag{48}$$

We introduce the following subset of $L^2(\Omega)$:

$$\mathcal{T}_{t_0} = \left\{\exp\left(\alpha X_{t_0} + \int_{t_0}^T h_s dX_s\right) \ ;\ \ \alpha \in \mathbb{R},\ h \in L^2(0,T)\right\}.$$

It is easy to check that $\mathcal{T}_{t_0}$ is a total subset of $L^2(\Omega, \mathcal{G}_{t_0}, \mathbb{Q})$.

Then, in order to compute the conditional expectation (48), we have to compute

$$\lim_{h\to 0^+} \frac{1}{h} E\left(F(X_{t_0} - X_{t_0-h})\right)$$

for any random variable F in $\mathcal{T}_{t_0}$. It is straightforward that for all $t \leqslant t_0$, $D_{t_0}F = D_t F$.

We now write for all square integrable deterministic function h_t:

$$\begin{aligned}
E\left[F\int_0^T h_t dX_t\right] &= E_{\mathbb{Q}}\left[G^{-1}F\int_0^T h_t dX_t\right]\\
&= E_{\mathbb{Q}}\left[G^{-1}\int_0^T D_tF.h_t dt\right] + E_{\mathbb{Q}}\left[F\int_0^T D_t(G^{-1}).h_t dt\right]\\
&= E\left[\int_0^T D_tF.h_t dt\right] + E\left[GF\int_0^T D_t(G^{-1}).h_t dt\right].
\end{aligned}$$

using $G^{-1} = \exp\left(\int_0^T b_s dX_s - 1/2\int_0^T b_s^2 ds\right)$ and the commutativity relationship between the Malliavin derivative and the stochastic integral (see (13), p.38) yield

$$\begin{aligned}
G.D_t(G^{-1}) &= b_t + \int_t^T D_t b_s dX_s - \int_0^T b_s.D_t b_s ds\\
&= b_t + \int_t^T D_t b_s dW_s.
\end{aligned}$$

Taking $h_t = \mathbf{1}_{[t_0-h,t_0]}(t)$, we thus obtain:

$$E[F(X_{t_0} - X_{t_0-h})] = E\left[\int_{t_0-h}^{t_0} D_tF dt\right] + E\left[F\int_{t_0-h}^{t_0}\left(b_t + \int_t^T D_t b_s dW_s\right)dt\right].$$

Since $F \in \mathcal{T}_{t_0}$, we have

$$E[F(X_{t_0} - X_{t_0-h})] = hE\left[D_{t_0}F\right] + E\left[F\int_{t_0-h}^{t_0}\left(b_t + \int_t^T D_t b_s dW_s\right)dt\right]. \tag{49}$$

With

$$E\left[\widehat{b}_{T-t_0}F\right] = -\lim_{h\to 0^+}\frac{1}{h}E[F(X_{t_0}-X_{t_0-h})]\ ,$$

we deduce that

$$-E\left[\widehat{b}_{T-t_0}F\right] = E\left[D_{t_0}F\right] + E\left[F\left(b_{t_0}+\int_{t_0}^T D_{t_0}b_s dW_s\right)\right]. \tag{50}$$

From (49), we write:

$$\begin{aligned} E\left[D_{t_0}F\right] &= \frac{1}{t_0}E\left[F\left(X_{t_0}-\int_0^{t_0}\left(b_t+\int_t^T D_t b_s dW_s\right)dt\right)\right] \\ &= \frac{1}{t_0}E\left[F\left(W_{t_0}-\int_0^{t_0}\int_t^T D_t b_s dW_s dt\right)\right]. \end{aligned}$$

Using (49), we conclude that:

$$\begin{aligned} -E\left[(\widehat{b}_{T-t_0}+b_{t_0})F\right] = E\Bigg[F\Bigg(&\int_{t_0}^T D_{t_0}b_s dW_s \\ &+\frac{1}{t_0}\left(W_{t_0}-\int_0^{t_0}\int_t^T D_t b_s dW_s dt\right)\Bigg)\Bigg]\ , \end{aligned}$$

and the formula for the reversed drift (21) is proved. □

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